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Generalized Vector Equilibrium Problems with Trifunctions

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Abstract. In this paper, we study the existence of strong and weak solutions of the generalized vector equilibrium problems for trifunctions. Two special classes of vector-valued trifunctions are introdcued, called the classes of (SPM) and (GPM), respectively. Some existence results for strong solutions associated to functions of these classes are given.

Key words: Generalized vector equilibrium problems, Topological pseudomonotonicity, (SPM) classes
and (GPM) classes

1. Introduction

Let 2^X denote the family of all subsets of a nonempty set X. For two topological spaces X and Y, a function from X into 2^Y will be called a *multi-valued mapping* of X into Y.

Let X and Y be real topological vector spaces, T be a multi-valued mapping of a set $K \subset X$ into a set $D \subset Y$, and let $f : K \times D \times K \longrightarrow \mathbb{R}$ be a real trifunction. The generalized equilibrium problem associated with (f, T, K, D) consists of finding

$$\widehat{x} \in K \quad \text{and} \quad \widehat{y} \in T(\widehat{x}) \quad \text{such that} \quad f(\widehat{x}, \widehat{y}, u) \geqslant 0 \quad \text{for all } u \in K.$$

Generalized equilibrium problems subsume in particular optimization problems, equilibrium problems (Bianchi and Schaible,1996; Blum and Oettli, 1994; Chadli *et al.*, 2000; Hadjisavvas and Schaible, 1996) and generalized variational inequalities. For instance, by considering the multi-valued mapping T of K into the topological dual X^* of X, and the trifunction $f(x, y, u) = \langle y, u - x\rangle$ for $(x, y, u) \in K \times X^* \times K$, the above generalized equilibrium problem becomes a generalized variational inequality which is to find

$$\widehat{x} \in K \quad \text{and} \quad \widehat{y} \in T(\widehat{x}) \quad \text{such that} \quad \langle \widehat{y}, u - \widehat{x}\rangle \geqslant 0 \quad \text{for all } u \in K.$$

Generalized vector equilibrium problems, (GVEP for short), are obtained from generalized equilibrium problems by considering trifunctions of $K \times D \times K$ into a real topological vector space $\mathcal{Z}$ with an ordering cone. By an ordering cone $C \subset \mathcal{Z}$

we mean that C is a closed convex cone in $\mathcal{Z}$ with Int$C \neq \emptyset$ and $C \neq \mathcal{Z}$, where Int$C$ denotes the interior of C.

Let K, D and T be given as above, and let $f: K \times D \times K \longrightarrow \mathcal{Z}$ be a trifunction. The problem (GVEP) is to find a pair $(\widehat{x}, \widehat{y}) \in K \times T(\widehat{x})$ such that

$$f(\widehat{x}, \widehat{y}, u) \in (-\mathrm{Int}C)^c \quad \text{for all } u \in K,$$

where $(-\mathrm{Int}C)^c$ is the complement of $-\mathrm{Int}C$ in $\mathcal{Z}$. Such an $\widehat{x}$ will be called a *strong solution* of the problem (GVEP) in the sense that $\widehat{y}$ does not depend on $u \in K$. While an $\widehat{x} \in K$ is called a *weak solution* of the problem (GVEP) if for every $u \in K$ there exists $\widehat{y} \in T(\widehat{x})$ (depending on u) such that $f(\widehat{x}, \widehat{y}, u) \in (-\mathrm{Int}C)^c$.

Similar to generalized variational inequalities, generalized vector variational inequalities are obtained from the problems (GVEP) by considering the multi-valued mappings of K into the space $L(X, \mathcal{Z})$ of all continuous linear mappings from X into $\mathcal{Z}$. More precisely, let T be a multi-valued mapping of K into $L(X, \mathcal{Z})$. The generalized vector variational inequality associated with (T, K) is to find

$$\widehat{x} \in K \quad \text{and} \quad \widehat{y} \in T(\widehat{x}) \quad \text{such that} \quad \langle \widehat{y}, u - \widehat{x} \rangle \in (-\mathrm{Int}C)^c \quad \text{for all } u \in K.$$

Existence of weak solutions of generalized vector variational inequalities was investigated by many authors. See for instances Ansari (1999), Chen and Craven (1990), Lee et al. (1997) and Lin et al. (1997). Very few papers have appeared in the literature on the existence of strong solutions of generalized vector variational inequalities, which are included as a particular case of our general problem (GVEP) (see Konnov and Yao, 1997; Hadjisavvas and Schaible, 1998; Ansari and Yao, 1999 a,b).

In this paper, we shall establish some existence results for weak and strong solutions of the problem (GVEP) associated with trifunctions which satisfy some properties that extend naturally to the vector framework of some well known concepts in nonlinear analysis. The rest of the paper is organized as follows.

In Section 2, we recall some terminology and some preliminary results that we shall need in the sequels. A generalized Fan-KKM theorem due to Shioji (1991, Theorem 3) is stated, and a brief discussion on a scalarization procedure due to Oettli (1997) is given.

In Section 3, we consider the problem (GVEP) associated with trifunctions which satisfy some properties that extend naturally to the vector framework of some well known concepts in nonlinear analysis, and prove our main existence results (Theorems 3.1–3.3) by use of Shioji's generalized Fan-KKM Theorem and Oettli's scalarization procedure. Theorems 3.1 and 3.3 are existence results for strong solutions, and Theorem 3.2 is for weak solutions.

In the final section, we introduce two classes of trifunctions which will be called the classes of (SPM) and (GPM), respectively. These two classes of trifunctions are vector-valued, and generalize the notion of topological pseudomonotonicity for real valued functions due to Brézis and Browder. Also, by using the main results obtained in Section 3, we prove some existence results for strong solutions of the problem (GVEP) associated with trifunctions of the above two classes.

We shall use the following notation. For any subset A of a topological space X, let A^c denote the complement of A in X, and let $\overline{A}$ denote the closure of A in X. If X is a topological vector space, we denote by $\mathrm{co}(A)$ the convex hull of A.

2. Preliminaries

In this section, we start with the definition of upper semicontinuous multi-valued mappings due to Berge (1963).

A multi-valued mapping T of a topological space X into another Y is called *upper semicontinuous* at $x \in X$ if for every open set V containing $T(x)$, there is an open set U containing x such that $T(u) \subset V$ for all $u \in U$. While T is called upper semicontinuous on X if T is upper semicontinuous at every $x \in X$, and if *$T(x)$ is compact for every $x \in X$.*

We remark that in the rest of this paper any multi-valued mapping which is upper semicontinuous on its domain always assumes compact values.

THEOREM 2.1. **(Berge, 1963, pp. 109–112).** *Let X and Y be two topological spaces, and let T be a multi-valued mapping of X into Y. If T is upper semicontinuous, then*

(i) *T is closed, that is, the graph $\{(x,y) \in X \times Y : y \in T(x)\}$ of T is closed in $X \times Y$, and*
(ii) *for every compact set $K \subset X$, the set $T(K) = \bigcup_{x \in K} T(x)$ is compact in Y.*

Next, we consider functions of a topological space into an ordered real topological vector space. Throughout this section and the sequels, let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, (see Section 1).

A function f of a topological space X into $\mathcal{Z}$ is called *C-upper semicontinuous* if for every $z \in \mathcal{Z}$ the set $f^{-1}(z - \mathrm{Int}C)$ is open in X, (see Tanaka, 1997). Tanaka proved in that a function f of a topological space X into $\mathcal{Z}$ is C-upper semicontinuous if and only if for every $x \in X$ and for every $w \in \mathrm{Int}C$, there is an open neighborhood $U = U(x)$ of x such that $f(u) \in f(x) + w - \mathrm{Int}C$ for all $u \in U$.

By a standard argument, we can prove easily the following proposition, and hence the proof will be omitted.

PROPOSITION 2.2. *Let X be a Hausdorff topological space, and let f be a function of X into $\mathcal{Z}$. Then f is C-upper semicontinuous on X if and only if for every $x \in X$, for every $w \in IntC$, and for any net $\{x_\alpha\}_{\alpha \in I}$ in X converging to x, there is an $\alpha_0 \in I$ such that*

$$\overline{\{f(x_\beta) : \beta \geqslant \alpha\}} \subset f(x) + w - IntC \quad \text{for all } \alpha \geqslant \alpha_0.$$

Let K be a nonempty convex subset of a real topological vector space X.

(i) A function $\varphi: K \longrightarrow \mathcal{Z}$ is called *C-convex* if

$$t\varphi(x)+(1-t)\varphi(x')-\varphi(tx+(1-t)x')\in \mathrm{Int}C\cup\{0\}$$

whenever x, $x'\in K$ and $0\leqslant t\leqslant 1$. Moreover, φ is called *C-concave* if $-\varphi$ is C-convex.

(ii) A bifunction $h: K\times K \longrightarrow \mathcal{Z}$ is called *C-quasiconvex-like*, (see Ansari and Yao, 1999), if

$$h(x, ty_1+(1-t)y_2)\in h(x, y_1)-C \quad \text{or} \quad h(x, ty_1+(1-t)y_2)\in h(x, y_2)-C$$

for all x, y_1, $y_2\in K$ and for $0\leqslant t\leqslant 1$.

(iii) A bifunction $h: K\times K \longrightarrow \mathcal{Z}$ is called *vector* 0*-diagonally convex*, (see Chadli *et al.*, 2002), if for any finite set $\{y_1, \ldots, y_n\}\subset K$,

$$\sum_{j=1}^{n} t_j h(x, y_j)\in(-\mathrm{Int}C)^c$$

whenever $x=\sum_{j=1}^{n} t_j y_j$ with $t_j\geqslant 0$ for all j and $\sum_{j=1}^{n} t_j=1$.

(iv) Let $T: K\longrightarrow 2^D$ be a multi-valued mapping. A trifunction $f: K\times D\times K\longrightarrow \mathcal{Z}$ is called *vector* 0*-diagonally convex with respect to* T if for any finite set $\{x_1, \ldots, x_n\}\subset K$ and for any $x=\sum_{j=1}^{n} t_j x_j$ with $t_j\geqslant 0$ and $\sum_{j=1}^{n} t_j=1$, there exists $y\in T(x)$ such that $\sum_{i=1}^{n} t_j f(x, y, x_j)\in(-\mathrm{Int}C)^c$.

REMARK 2.1. Let K, D and T be given as above.

(i) If $h: K\times K\longrightarrow \mathcal{Z}$ is a vector 0-diagonally convex bifunction, then $h(x, x)\in(-IntC)^c$ for every $x\in K$ since $\{x\}$ is a convex subset of K.

(ii) Let $f: K\times D\times K\longrightarrow \mathcal{Z}$ be a trifunction such that for every $x\in K$ there exists $y_x\in T(x)$ satisfying $f(x, y_x, x)=0$. If for every fixed $(x, y)\in K\times D$, the function $u\longmapsto f(x, y, u)$ is C-convex, then f is vector 0-diagonally convex with respect to T.

By a convexity argument, we obtain the following proposition easily.

PROPOSITION 2.3. *Let K be a nonempty convex subset of a Hausdorff real topological vector space, let $h: K\times K \longrightarrow \mathcal{Z}$ be a bifunction, and let E be any nonempty finite subset of K.*

(i) *If h is C-quasiconvex-like, and if there exists $\widehat{y}\in co(E)$ such that $h(x,\widehat{y})\in(-IntC)^c$, then $h(x,y)\in(-IntC)^c$ for some $y\in E$.*

(ii) *If h is vector 0-diagonally convex, and if $x\in co(E)$, then $h(x,y)\in(-IntC)^c$ for some $y\in E$.*

Finally, we shall state Shioji's generalized Fan-KKM theorem (Shioji, 1991, Theorem 3), and give a brief discussion on Oettli's scalarization procedure.

Let K be a nonempty convex subset of a topological vector space X, and let G and H be two multi-valued mappings from K into a topological space Y. The mapping G is called an H-KKM mapping if

$$H(\mathrm{co}(E))\subset \bigcup_{x\in E} G(x)$$

for every nonempty finite set $E\subset K$.

THEOREM 2.4. **(Shioji)** *Let K be a nonempty compact and convex subset of a topological vector space X, and let G and H be two multi-valued mappings from K into a topological vector space Y. If*

(i) *H is upper semicontinuous and $H(x)$ is nonempty and convex for every $x\in K$;*
(ii) *G is an H-KKM mapping and $G(x)$ is closed for every $x\in K$,*

then $\bigcap_{x\in K} g(x)\neq\emptyset$.

Let $\mathcal{Z}^*$ be the topological dual of $\mathcal{Z}$. The set

$$C^*=\{z^*\in\mathcal{Z}^*:\langle z^*,z\rangle\geqslant 0 \ \text{ for all } z\in C\}$$

is the polar cone of C. Note that C^* has a weak * compact base B, i.e., $B\subset C^*$ is convex and weak* compact with $0\notin B$ and $C^*=\bigcup_{t>0} tB$, (see Luc, 1989). Let $\psi:\mathcal{Z}\longrightarrow\mathbb{R}$ be defined by

$$\psi(z)=\max_{\lambda\in B}\langle\lambda,z\rangle \qquad \text{for } z\in\mathcal{Z}.$$

Then ψ is sublinear, hence convex and lower semicontinuous. Also, for $z\in\mathcal{Z}$, we have

$$z\in(-\mathrm{Int}C)^c\Longleftrightarrow\psi(z)\geqslant 0. \quad \text{(See, e.g., Oettli, 1997)}$$

PROPOSITION 2.5. *Let f be a function from a real Hausdorff topological vector space X into $\mathcal{Z}$, and let $g: X \longrightarrow \mathbb{R}$ be defined by $g(x) = \psi(f(x))$ for $x \in X$.*

(i) *If f is C-upper semicontinuous on X, then g is upper semicontinuous.*
(ii) *If f is C-convex (respectively, C-concave), then g is convex (respectively, concave).*

Proof. The statement (ii) follows immediately from the definition. The statement (i) is proved as follows.

Since f is C-upper semicontinuous on X, then by Proposition 2.1, for each $w \in \operatorname{Int} C$ and for any net $\{x_\alpha\}_{\alpha \in I}$ in X converging to x there exists an $\alpha_0 \in I$ such that

$$\overline{\{f(x_\beta): \beta \geqslant \alpha\}} \subset f(x) + w - \operatorname{Int} C \quad \text{for all } \alpha \geqslant \alpha_0.$$

Choose an arbitrary net $\{w_j\}_{j \in J}$ in $\operatorname{Int} C$ with $w_j \to 0$. For each $j \in J$, there is an $\alpha_0(j) \in I$ such that

$$f(x_\beta) - f(x) - w_j \in (-\operatorname{Int} C) \ \text{ for all } \beta \geqslant \alpha_0(j).$$

Then, for all $\beta \geqslant \alpha_0(j)$, we have $\psi(f(x_\beta) - f(x) - w_j) < 0$ and

$$g(x_\beta) = \psi(f(x_\beta)) \leqslant \psi(f(x)) + \psi(w_j) = g(x) + \psi(w_j).$$

This implies that $\limsup_\beta g(x_\beta) \leqslant g(x)$ since $w_j \to 0$. The proof is complete.

PROPOSITION 2.6. **(Blum and Oettli, 1994, Lemma 1).** *Let X and Y be two real Hausdorff topological vector spaces, let $K \subset X$ be convex and compact, and let $D \subset Y$ be convex. Let $p: K \times D \longrightarrow \mathbb{R}$ be a bifunction. Assume that*

(i) *for every $y \in D$, the function $x \longmapsto p(x, y)$ is concave and upper semicontinuous, and*
(ii) *for every $x \in K$, the function $y \longmapsto p(x, y)$ is convex.*

If $\max_{x \in K} p(x, y) \geqslant 0$ for all $y \in D$, then there exists $\widehat{x} \in K$ such that $p(\widehat{x}, y) \geqslant 0$ for all $y \in D$.

3. Existence Results

In this section, we state and prove our main existence results for the problem (GVEP). The first result is obtained by using Shioji's Fan-KKM Theorem.

THEOREM 3.1. Let X and Y be two real Hausdorff topological vector spaces, let $D \subset Y$ be nonempty and closed, and let $K \subset X$ be nonempty convex. Let $T: K \longrightarrow 2^D$ be a multi-valued mapping, and let $f: K \times D \times K \longrightarrow \mathcal{Z}$ be a trifunction. Assume that the following conditions are satisfied.

(i) $f(x,y,x)\in(-\text{Int}C)^c$ for all $x\in K$ and $y\in T(x)$.

(ii) T is upper semicontinuous, and $T(x)$ is nonempty and convex for every $x\in K$.

(iii) For every $y\in D$, the bifunction $(x,u)\longmapsto f(x,y,u)$ is C-quasiconvex-like *(*or vector 0-diagonally convex*)* on $K\times K$.

(iv) For every $u\in K$, the bifunction $(x,y)\longmapsto f(x,y,u)$ is C-upper semicontinuous on $\text{co}(E)\times D$ for every nonempty finite set $E\subset K$.

(v) For any $u\in K$ and for any net $\{(x_\alpha,y_\alpha)\}$ in $K\times D$ converging to $(x,y)\in K\times D$, if

$$f(x_\alpha,y_\alpha,tu+(1-t)x)\in(-\text{Int}C)^c$$

for all α and for $0\leqslant t\leqslant 1$, then $f(x,y,u)\in(-\text{Int}C)^c$.

(vi) *(Coercivity)* There is a nonempty compact set $K_0\subset K$, and there is a nonempty compact convex set $K_1\subset K$ such that

$$x\in K\cap K_0^c\Longrightarrow f(x,y,u_x)\in-\text{Int}C \quad \text{for some } u_x\in K_1 \text{ and for all } y\in T(x).$$

Then there is a pair $(\widehat{x},\widehat{y})\in K\times T\widehat{x})$ such that $f(\widehat{x},\widehat{y},u)\in(-\text{Int}C)^c$ for all $u\in K$.

REMARK 3.1. In Theorem 3.1,

(i) if for every $y\in D$ the bifunction $(x,u)\longmapsto f(x,y,u)$ is vector 0-diagonally convex on $K\times K$, then condition (i) is redundant, (cf. Remark 2.1 (i));
(ii) if K is compact, the coercivity condition (vi) can be deleted.

We start the proof of Theorem 3.1 by showing that the following lemma holds.

LEMMA 3.1. *Let $K\subset X$, $D\subset Y$, T and f be given as in Theorem* 3.1. *If E is any nonempty finite subset of K, then there exist $\widehat{x}\in co(E)$ and $\widehat{y}\in T(\widehat{x})$ such that $f(\widehat{x},\widehat{y},u)\in(-IntC)^c$ for all $u\in co(E)$.*

Proof. Let $G:\text{co}(E)\longrightarrow 2^{\text{co}(E)}\times 2^D$ and $H:\text{co}(E)\longrightarrow 2^{\text{co}(E)}\times 2^D$ be multi-valued mappings defined by

$$G(u)=\{(x,y)\in\text{co}(E)\times T(x):f(x,y,u)\in(-\text{Int}C)^c\} \quad\text{and}\quad H(x)=\{x\}\times T(x).$$

Note that by condition (ii), H is upper semicontinuous, and $H(x)$ is convex and compact for every $x\in\text{co}(E)$.

We shall prove that G is an H-KKM mapping, and that $G(u)$ is closed in $X\times Y$ for every $u\in\mathrm{co}(E)$. By Theorem 2.2, we then conclude $\bigcap_{u\in\mathrm{co}(E)} g(u)\neq\emptyset$. This completes the proof.

For every $u\in K$, let $f_u: K\times D\longrightarrow\mathcal{Z}$ be defined by

$$f_u(x,y)=f(x,y,u)\quad \text{for } (x,y)\in K\times D.$$

Since f_u is C-upper semicontinuous on $\mathrm{co}(E)\times D$, and since $H(\mathrm{co}(E))$ is compact in $X\times D$, for every $u\in\mathrm{co}(E)$, the set

$$G(u)=H(\mathrm{co}(E))\cap f_u^{-1}(-\mathrm{Int}C)^c$$

is closed in $H(\mathrm{co}(E))$, and hence it is closed in $X\times Y$ since D is closed in Y.

Finally, we prove that G is an H-KKM mapping. For any nonempty finite subset A of $\mathrm{co}(E)$, let $x\in\mathrm{co}(A)$ and $y\in T(x)$. Since $f(x,y,x)\in(-\mathrm{Int}C)^c$, by condition (iii) and Proposition 2.2, there exists $u\in A$ such that $f(x,y,u)\in(-\mathrm{Int}C)^c$. This proves that $(x,y)\in G(u)$, and that $H(\mathrm{co}(A))\subset\bigcup_{u\in A} g(u)$. Therefore, G is an H-KKM mapping.

Proof of Theorem 3.1. We first prove the theorem for the case where K is compact. Let $\mathcal{F}$ be the family of all nonempty finite subsets of K, and for every $E\in\mathcal{F}$ let

$$M_E=\{(x,y)\in K\times T(x): f(x,y,u)\in(-\mathrm{Int}C)^c \text{ for all } u\in\mathrm{co}(E)\}.$$

It follows from Lemma 3.1 that every M_E is nonempty.

We claim that $\bigcap_{E\in\mathcal{F}}\overline{M_E}\neq\emptyset$. Note that $T(K)$ is compact by Theorem 2.1 (ii). By definition, one proves easily that

$$E,F\in\mathcal{F}\Longrightarrow M_{E\cup F}\subset M_E\cap M_F \quad\text{and}\quad \overline{M_{E\cup F}}\subset\overline{M_E}\cap\overline{M_F}.$$

This implies that the family $\{\overline{M_E}: E\in\mathcal{F}\}$ possesses the finite intersection property. Since $M_E\subset K\times T(K)$, and since $K\times T(K)$ is compact, the claim is valid.

Let $(\widehat{x},\widehat{y})\in\bigcap_{E\in\mathcal{F}}\overline{M_E}$. We shall prove that $\widehat{y}\in T(\widehat{x})$ and $f(\widehat{x},\widehat{y},u)\in(-\mathrm{Int}C)^c$ for all $u\in K$. For every fixed $u\in K$, we consider the set $E_u=\{\widehat{x},u\}\subset K$. Since $(\widehat{x},\widehat{y})\in\overline{M_{E_u}}$, there is a net $\{(x_\alpha,y_\alpha)\}$ in M_{E_u} such that $\lim_\alpha(x_\alpha,y_\alpha)=(\widehat{x},\widehat{y})$. Note that $\widehat{y}\in T(\widehat{x})$ since T is closed. By the definition of (x_α,y_α), we have

$$f(x_\alpha,y_\alpha,tu+(1-t)\widehat{x})\in(-\mathrm{Int}C)^c\quad \text{for all } \alpha \text{ and for } 0\leqslant t\leqslant 1.$$

It follows immedately from condition (v) that $f(\widehat{x},\widehat{y},u)\in(-\mathrm{Int}C)^c$. Therefore, the theorem holds when K is compact.

Now, we complete the proof as follows. For every $E\in\mathcal{F}$, let

$$N_E=\{(x,y)\in K_0\times T(x): f(x,y,u)\in(-\mathrm{Int}C)^c \quad \text{for all } u\in \mathrm{co}(K_1\cup E)\}.$$

Note that $\mathrm{co}(K_1\cup E)$ is compact for every $E\in\mathcal{F}$ (see Aliprantis and Border, 1994, Lemma 4.12, p. 126). From previous discussion, there is a pair $(x_E,y_E)\in \mathrm{co}(K_1\cup E)\times T(x_E)$ such that $f(x_E,y_E,u)\in(-\mathrm{Int}C)^c$ for all $u\in\mathrm{co}(K_1\cup E)$. Since $K_1\subset \mathrm{co}(K_1\cup E)$, $x_E\in K_0$ by condition (vi). This proves that $N_E\neq\emptyset$ for every $E\in\mathcal{F}$.

By the compactness of $K_0\times T(K_0)$, and the same argument as above, we obtain $\bigcap_{E\in\mathcal{F}}\overline{N_E}\neq\emptyset$. Let $(\widehat{x},\widehat{y})\in\bigcap_{E\in\mathcal{F}}\overline{N_E}$. We claim that $(\widehat{x},\widehat{y})$ is a required solution.

For any fixed $u\in K$, let $E_u=\{\widehat{x},u\}$. Since $(\widehat{x},\widehat{y})\in\overline{N_{E_u}}$, there is a net $\{(x_\alpha,y_\alpha)\}$ in N_{E_u} converging to $(\widehat{x},\widehat{y})$. Since $\mathrm{co}(E_u)\subset\mathrm{co}(K_1\cup E_u)$,

$$f(x_\alpha,y_\alpha,\lambda u+(1-\lambda)\widehat{x})\in(-\mathrm{Int}C)^c \qquad \text{for all } \alpha \text{ and for } 0\leqslant\lambda\leqslant 1.$$

It follows from condition (v) that $f(\widehat{x},\widehat{y},u)\in(-\mathrm{Int}C)^c$.

Next, we use Oettli's scalarization procedure to derive two existence results for the problem (GVEP), stated as Theorem 3.2 for weak solutions and Theorem 3.3 for strong solutions, respectively.

THEOREM 3.2. Let X and Y be real Hausdorff topological vector spaces, let $D\subset Y$ be nonempty, and let $K\subset X$ be nonempty and convex. Let $T:K\longrightarrow 2^D$ be a multi-valued mapping, and let $f:K\times D\times K\longrightarrow\mathcal{Z}$ be a trifunction. Assume that the following conditions are satisfied.

(i) T is upper semicontinuous, and $T(x)$ is nonempty convex for every $x\in K$.

(ii) For every $x\in K$, there exists $y\in T(x)$ such that $f(x,y,x)=0$.

(iii) For each $(x,y)\in K\times D$, the function $u\longmapsto f(x,y,u)$ is C-convex.

(iv) For each fixed $u\in K$, the bifunction $(x,y)\longmapsto f(x,y,u)$ is C-upper semicontinuous on $K\times D$.

(v) For each $(x,u)\in K\times K$, the function $y\longmapsto f(x,y,u)$ is C-concave.

(vi) For every $u\in K$ and any net $\{(x_\alpha,y_\alpha)\}_{\alpha\in I}$ in $K\times D$ with $y_\alpha\in T(x_\alpha)$, $x_\alpha\to x\in K$, and

$$f(x_\alpha,y_\alpha,tu+(1-t)x)\in(-\mathrm{Int}C)^c \text{ for all } \alpha\in I \text{ and } 0\leqslant t\leqslant 1,$$

there exists $y_u \in T(x)$ such that $f(x, y_u, u) \in (-\text{Int}C)^c$.

(vii) There is a nonempty compact set $K_0 \subset K$, and there is a nonempty compact convex set $K_1 \subset K$ such that if $x \in K \cap K_0^c$, then $f(x, y, u_x) \in (-\text{Int}C)$ for some $u_x \in K_1$ and for all $y \in T(x)$.

Then there exists $\widehat{x} \in K$ such that for every fixed $u \in K$ there exists $\widehat{y}_u \in T(\widehat{x})$ satisfying $f(\widehat{x}, \widehat{y}_u, u) \in (-\text{Int}C)^c$.

THEOREM 3.3. Let $K \subset X$, $D \subset Y$, T and f be given in Theorem 3.2. If, in addition,

(viii) for each $(x, u) \in K \times K$, the function $y \longmapsto f(x, y, u)$ is C-upper semicontinuous,

then there exists $(\widehat{x}, \widehat{y}) \in K \times T(\widehat{x})$ such that $f(\widehat{x}, \widehat{y}, u) \in (-\text{Int}C)^c$ for all $u \in K$.

To derive Theorem 3.2 and 3.3, we first establish Lemma 3.2 and 3.3.

LEMMA 3.2. *Let X and Y be real Hausdorff topological vector spaces, let $K \subset X$ be nonempty and convex, and let $D \subset Y$ be nonempty. Let $T : K \longrightarrow 2^D$ be an upper semicontinuous multi-valued mapping, and let $f : K \times D \times K \longrightarrow \mathcal{Z}$ be a trifunction. Let E be any nonempty finite subset of K. If*

(*i*) *f is vector 0-diagonally convex with respect to T, and*
(*ii*) *for every fixed $u \in K$, the bifunction $(x, y) \longmapsto f(x, y, u)$ is C-upper semicontinuous on $co(E) \times D$,*

then there is an $\widehat{x} \in co(E)$ such that for every fixed $u \in co(E)$ there exists $y_u \in T(\widehat{x})$ with $f(\widehat{x}, y_u, u) \in (-IntC)^c$.

Proof. For every $u \in \text{co}(E)$, let

$$S(u) = \{x \in \text{co}(E) : f(x, y, u) \in (-\text{Int}C)^c \quad \text{for some } y \in T(x)\}.$$

We prove that $\bigcap_{u \in \text{co}(E)} S(u) \neq \emptyset$ by using the Ky Fan Lemma (Fan, 1961). This completes the proof.

It follows from (i) that $u \in S(u)$ for every $u \in \text{co}(E)$. Thus $S(u)$ is nonempty for every $u \in \text{co}(E)$. To apply the Ky Fan Lemma, we have to show that every $S(u)$ is closed. Consequently, every $S(u)$ is compact.

Let $\{x_\alpha\}$ be any net in $S(u)$ converging to x. We prove that $x \in S(u)$. For each α, there exists $y_\alpha \in T(x_\alpha)$ such that $f(x_\alpha, y_\alpha, u) \in (-\text{Int}C)^c$. Since co$(E)$ is compact, and since T is upper semicontinuous on K, $T(\text{co}(E))$ is compact, and without loss of generality, we can assume that $y_\alpha \to y \in T(\text{co}(E))$. Note that $y \in T(x)$ since T is closed. From condition (ii), we conclude $f(x, y, u) \in (-\text{Int}C)^c$ and $x \in S(u)$.

For any finite set $A = \{x_1, ..., x_n\} \subset \text{co}(E)$, we write any $x \in \text{co}(A)$ as

$$x = \sum_{j=1}^{n} t_j x_j \quad \text{with } t_j \geqslant 0 \text{ and } \sum_{j=1}^{n} t_j = 1.$$

Since f is vector 0-diagonally convex with respect to T, there exists $y \in T(x)$ such that $\sum_{j=1}^{n} t_j f(x, y, x_j) \in (-\text{Int}C)^c$. Then $f(x, y, x_j) \in (-\text{Int}C)^c$ for some j with $1 \leqslant j \leqslant n$, and $x \in S(x_j)$. This proves that $\text{co}(A) \subset \bigcup_{j=1}^{n} S(x_j)$. Therefore, $\bigcap_{u \in \text{co}(E)} S(u) \neq \emptyset$.

LEMMA 3.3. *Let X and Y be real Hausdorff topological vector spaces, let $K \subset X$ be nonempty and convex, and let $D \subset Y$ be nonempty. Let $T: K \longrightarrow 2^D$ be a multi-valued mapping, and let $f: K \times D \times K \longrightarrow \mathcal{Z}$ be a trifunction. Let E be any nonempty finite subset of K. Assume that the following conditions are satisfied.*

(i) *T is upper semicontinuous, and $T(x)$ is nonempty convex for every $x \in K$.*

(ii) *For every $x \in K$, there exists $y \in T(x)$ such that $f(x, y, x) = 0$.*

(iii) *For each $(x, y) \in K \times D$, the function $u \longmapsto f(x, y, u)$ is C-convex.*

(iv) *For each $(x, u) \in K \times K$, the function $y \mapsto f(x, y, u)$ is C-concave.*

(v) *For every fixed $u \in K$, the bifunction $(x, y) \longmapsto f(x, y, u)$ is C-upper semicontinuous on $co(E) \times D$.*

Then there is a pair $(\widehat{x}, \widehat{y}) \in co(E) \times T(\widehat{x})$ such that $f(\widehat{x}, \widehat{y}, u) \in (-IntC)^c$ for all $u \in co(E)$.

Proof. First, note that f is vector 0-diagonally convex with respect to T, (See Remark 2.1 (ii).). By Lemma 3.2, there is an $\widehat{x} \in \text{co}(E)$ such that for every fixed $u \in \text{co}(E)$ there exists $y_u \in T(\widehat{x})$ with $f(\widehat{x}, y_u, u) \in (-\text{Int}C)^c$.

To use the scalarization method, let $\varphi(y,u)=\psi(f(\widehat{x},y,u))$ for $(y,u)\in D\times K$. From Proposition 2.3, and conditions (iii), (iv) as well as (v), we know that

(a) for every $u\in K$, the function $y\longmapsto\varphi(y,u)$ is upper semicontinuous and concave, and
(b) for every $y\in D$, the function $u\longmapsto\varphi(y,u)$ is convex.

Since for every $u\in \mathrm{co}(E)$,

$$\max_{y\in T(\widehat{x})}\varphi(y,u)\geqslant\varphi(y_u,u)=\psi(f(\widehat{x},y_u,u))\geqslant 0,$$

by Proposition 2.4, there exists $\widehat{y}\in T(\widehat{x})$ such that $\varphi(\widehat{y},u)\geqslant 0$ for all $u\in\mathrm{co}(E)$. Therefore, $f(\widehat{x},\widehat{y},u)\in(-\mathrm{Int}C)^c$ for all $u\in\mathrm{co}(E)$.

Proof of Theorem 3.2. First, we assume that K is compact. Let $\mathcal{F}$ be the family of all nonempty finite subsets of K. For every $E\in\mathcal{F}$, let

$$M_E=\{x\in K:\text{there exists y}\in T(x)\text{such that}$$
$$f(x,y,u)\in(-\mathrm{Int}C)^c\text{for all}u\in\mathrm{co}(E)\}.$$

It follows from Lemma 3.3 that $M_E\neq\emptyset$ for all $E\in\mathcal{F}$. Now, by the same reasoning as that in the proof of Theorem 3.1, we obtain $\bigcap_{E\in\mathcal{F}}\overline{M_E}\neq\emptyset$.

Let $\widehat{x}\in\bigcap_{E\in\mathcal{F}}\overline{M_E}$. For $u\in K$, consider the set $E=\{u,\widehat{x}\}$. Since $\widehat{x}\in\overline{M_{E_u}}$, there is a net $\{x_\alpha\}$ in M_{E_u} such that $x_\alpha\to\widehat{x}$, and there exists $y_\alpha\in T(x_\alpha)$ such that

$$f(x_\alpha,y_\alpha,tu+(1-t)\widehat{x})\in(-\mathrm{Int}C)^c\quad\text{ for all }0\leqslant t\leqslant 1.$$

By condition (vi), there exists $\widehat{y}_u\in T(\widehat{x})$ such that $f(\widehat{x},\widehat{y}_u,u)\in(-\mathrm{Int}C)^c$. This proves the theorem for the case where K is compact.

Now, we complete the proof of the theorem as follows. For every $u\in K$, let

$$S(u)=\{x\in K_0:\text{there exists }y_u\in T(x)\text{ such that }f(x,y_u,u)\in(-\mathrm{Int}C)^c\}.$$

We shall prove that every $S(u)$ is nonempty, and that the family $\{S(u)\}_{u\in K}$ has the finite intersection property.

For any finite set $\{u_1,\ldots,u_n\}\subset K$, we consider the set $B=\mathrm{co}(K_1\cup\{u_1,\ldots,u_n\})$. Since K_1 is compact and convex, B is compact. (See Bourbaki, 1996, Theorem 15, II.14.) By what has been proved above, there exists $\tilde{x}\in B$ such that for every fixed $u\in B$ there exists $\tilde{y}_u\in T(\tilde{x})$ such that $f(\tilde{x},\tilde{y}_u,u)\in(-\mathrm{Int}C)^c$. Since $K_1\subset B$,

$\tilde{x} \in K_0$ by the condition (vii). Thus $\tilde{x} \in \bigcap_{j=1}^{n} S(u_j)$. Therefore, $S(u)$ is nonempty for every $u \in K$, and the family $\{S(u)\}_{u \in K}$ has the finite intersection property.

It is not difficult to see that $S(u)$ is closed for each $u \in K$. As K_0 is compact, we have $\bigcap_{u \in K} S(u) \neq \emptyset$ and each $\widehat{x} \in \bigcap_{u \in K} S(u)$ is a weak solution of (GVEP). The proof is now complete.

Proof of Theorem 3.3. We shall prove the theorem by use of the scalarization procedure, and consider the function $\varphi(x,y,u) = \psi(f(x,y,u))$. From Proposition 2.3 and conditions (iii), (v) as well as (vii), we know that

(a) for every (x,y) the function $u \longmapsto \varphi(x,y,u)$ is convex, and
(b) for every (x,u) the function $y \longmapsto \varphi(x,y,u)$ is concave and upper semicontinuous.

By Theorem 3.2 and Remark 2.1 (ii), there exists $\widehat{x} \in K$ such that for each $u \in K$ there exists $\widehat{y}_u \in T(\widehat{x})$ with $f(\widehat{x}, \widehat{y}_u, u) \in (-\text{Int}C)^c$. Thus $\max_{y \in T(\widehat{x})} \varphi(\widehat{x}, y, u) \geqslant 0$ for all $u \in K$.

Now, we consider the function $p(y,u) = \varphi(\widehat{x}, y, u)$ for $(y,u) \in T(\widehat{x}) \times K$. Since $T(\widehat{x})$ is convex and compact, and since K is convex, then by Proposition 2.4, there exists $\widehat{y} \in T(\widehat{x})$ such that $\varphi(\widehat{x}, \widehat{y}, u) \geqslant 0$ for all $u \in K$. The proof is complete.

4. The Classes SPM and GPM

In this section, we shall introduce two classes of vector-valued trifunctions which generalize the notion of topological pseudomonotonicity for real functions introduced by Brézis (1968) and Browder (1976).

We start with the definition of superior and inferior of a subset of $\mathcal{Z}$. For a set $A \subset \mathcal{Z}$, the superior $\text{Sup}A$ and the inferior $\text{Inf}A$ of A are defined by

$$\text{Sup}A = \{z \in \overline{A} : A \cap (z + \text{Int}C) = \emptyset\}$$

and

$$\text{Inf}A = \{z \in \overline{A} : A \cap (z - \text{Int}C) = \emptyset\},$$

respectively. It follows immediately from the definition that $\mathrm{Sup}\overline{A} = \mathrm{Sup}A$ and $\mathrm{Inf}\overline{A} = \mathrm{Inf}A$.

If $\{z_\alpha\}_{\alpha\in I}$ is a net in $\mathcal{Z}$, then we define the limit inferior and limit superior of $\{z_\alpha\}$ by

$$\mathrm{Liminf}\, z_\alpha = \mathrm{Sup}\Big(\bigcup_{\alpha\in I}\mathrm{Inf}A_\alpha\Big) \quad \text{and} \quad \mathrm{Limsup}\, z_\alpha = \mathrm{Inf}\Big(\bigcup_{\alpha\in I}\mathrm{Sup}A_\alpha\Big),$$

where $A_\alpha = \{z_\beta : \beta \geqslant \alpha\}$ for every $\alpha \in I$.

Let X and Y be two Hausdorff topological spaces, let $f : X\times Y\times X \longrightarrow \mathcal{Z}$ be a trifunction, and let $T : X \longrightarrow 2^Y$ be a multi-valued mapping.

(1) f is said to be of class (SPM) if for every $w \in \mathrm{Int}C$ and for each net $\{(x_\alpha, y_\alpha)\}_{\alpha\in I}$ in $X\times Y$ satisfying

$$(x_\alpha, y_\alpha) \to (x, y) \in X\times Y \quad \text{and} \quad \mathrm{Liminf}\, f(x_\alpha, y_\alpha, x) \cap (-\mathrm{Int}C) = \emptyset,$$

there is an $\alpha_0 \in I$ such that

$$\overline{\{f(x_\beta, y_\beta, u) : \beta \geqslant \alpha\}} \subset f(x, y, u) + w - \mathrm{Int}C$$

for all $\alpha \geqslant \alpha_0$ and for all $u \in X$.
Consequently, $\mathrm{Sup}\{f(x_\beta, y_\beta, u) : \beta \geqslant \alpha\} \subset f(x, y, u) + w - \mathrm{Int}C$ for all $\alpha \geqslant \alpha_0$ and for all $u \in X$.

(2) f is said to be of class (GPM) with respect to T if it posseses the following property. For every $w \in \mathrm{Int}C$ and for each net $\{(x_\alpha, y_\alpha)\}_{\alpha\in I}$ in $X\times Y$ with $y_\alpha \in T(x_\alpha)$ and $x_\alpha \to x \in X$, if there exists $\gamma_0 \in I$ such that

$$\overline{\{f(x_\beta, y_\beta, x) : \beta \geqslant \alpha\}} \cap (-w - \mathrm{Int}C) = \emptyset \quad \text{for all } \alpha \geqslant \gamma_0,$$

then there exists $\alpha_0 \in I$ such that for each $u \in X$ there exists $y_u \in T(x)$ with

$$\overline{\{f(x_\beta, y_\beta, u) : \beta \geqslant \alpha\}} \subset f(x, y_u, u) - w - \mathrm{Int}C \quad \text{for all } \alpha \geqslant \alpha_0.$$

REMARK 4.1. Let K be a closed convex subset of a reflexive Banach space B. A multi-valued mapping $T : K \longrightarrow 2^{B^*}$ is called pseudomonotone in the sense of Browder (see Browder, 1976; Zeidler, 1985, p. 913) if it satisfies the following condition.

For any sequence $\{(u_n, u_n^*)\}_{n=1}^\infty$ in $K\times B^*$ with $u_n^* \in T(u_n)$ for all integers $n \geqslant 1$, and with $u_n \rightharpoonup u$ in $\sigma(B, B^*)$ as $n \to \infty$, if

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leqslant 0,$$

then for every $x\in K$, there exists $u_x^*\in B^*$ such that $u_x^*\in T(x)$ and

$$\langle u_x^*, u-x\rangle \leqslant \liminf_{n\to\infty}\langle u_n^*, u_n-x\rangle .$$

It is easy to see that topological pseudomonotonicity of the multi-valued mapping T in the sense of Browder implies that the trifunction f defined by $f(u,u^*,x)=\langle u^*, x-u\rangle$ for $(u,u^*,x)\in K\times D\times K$ is of class (GPM). Therefore, the above definition represents an extension to a vector framework of the classical pseudomonotonicity notion introduced by Brézis and Browder.

THEOREM 4.1. Let X and Y be real Hausdorff topological vector spaces, let $K\subset X$ be nonempty convex, and let $D\subset Y$ be nonempty closed. Let $T:K\longrightarrow 2^D$ be a multi-valued mapping, and let $f:K\times D\times K\longrightarrow \mathcal{Z}$ be a trifunction. Assume that the following conditions are satisfied.

(i) $f(x,y,x)\in(-\text{Int}C)^c$ for all $x\in K$ and $y\in T(x)$.

(ii) f is of class *(SPM)*.

(iii) T is upper semicontinuous, and $T(x)$ is nonempty and convex for every $x\in K$.

(iv) For every $y\in D$, the bifunction $(x,u)\longmapsto f(x,y,u)$ is C-quasiconvex-like *(*or vector 0-diagonally convex*)* on $K\times K$.

(v) For every $u\in K$, the bifunction $(x,y)\longmapsto f(x,y,u)$ is C-upper semicontinuous on $\text{co}(E)\times D$ for every nonempty finite set $E\subset K$.

(vi) *(Coercivity)* There is a nonempty compact set $K_0\subset K$, and there is a nonempty compact convex set $K_1\subset K$ such that

$$x\in K\cap K_0^c \Longrightarrow f(x,y,u_x)\in -\text{Int}C$$
$$\text{for some } u_x\in K_1 \text{ and for all } y\in T(x).$$

Then there exist $\widehat{x}\in K$ and $\widehat{y}\in T(\widehat{x})$ such that $f(\widehat{x},\widehat{y},u)\in(-\text{Int}C)^c$ for all $u\in K$.

Theorem 4.1 follows immediately from Theorem 3.1 and the following result.

LEMMA 4.1. Let X and Y be real Hausdorff topological vector spaces, let $K\subset X$ be nonempty convex, and let $D\subset Y$ be nonempty. Let $f:K\times D\times K\longrightarrow\mathcal{Z}$ be a trifunction of class *(SPM)*. If $u\in K$, and $\{(x_\alpha,y_\alpha)\}_{\alpha\in I}$ is a net in $K\times D$

converging to $(x, y) \in K \times D$ satisfying

$$f(x_\alpha, y_\alpha, tu+(1-t)x) \in (-\mathrm{Int}C)^c$$
$$\text{for all} \alpha \in I \text{and for all} 0 \leqslant t \leqslant 1,$$

then $f(x, y, u) \in (-\mathrm{Int}C)^c$.

Proof. For any $u \in K$ and for any $\alpha \in I$, let

$$A_\alpha(u) = \{f(x_\beta, y_\beta, u) : \beta \geqslant \alpha\}.$$

By assumption, $f(x_\alpha, y_\alpha, x) \in (-\mathrm{Int}C)^c$ and $f(x_\alpha, y_\alpha, u) \in (-\mathrm{Int}C)^c$ for all $\alpha \in I$. This implies that $A_\alpha(x) \subset (-\mathrm{Int}C)^c$ and $A_\alpha(u) \subset (-\mathrm{Int}C)^c$ for all $\alpha \in I$. Note that

$$\mathrm{Liminf} f(x_\alpha, y_\alpha, x) \cap (-\mathrm{Int}C) = \emptyset$$

since $A_\alpha(x) \subset (-\mathrm{Int}C)^c$ for all $\alpha \in I$.

Suppose that $f(x, y, u) = -w$ for some $w \in \mathrm{Int}C$. Since f is of class (SPM), there is an $\alpha_0 \in I$ such that

$$\alpha \geqslant \alpha_0 \Longrightarrow A_\alpha(u) \subset f(x, y, u) + w - \mathrm{Int}C = -\mathrm{Int}C.$$

This is a contradiction. Hence $f(x, y, u) \in (-\mathrm{Int}C)^c$.

COROLLARY 4.1. Let X and Y be real Hausdorff topological vector spaces, let $K \subset X$ be nonempty convex, and let $D \subset Y$ be nonempty closed. Let $T : K \longrightarrow 2^D$ be a multi-valued mapping, and let $f : K \times D \times K \longrightarrow \mathbb{R}$ be a trifunction. Assume that the following conditions are satisfied.

(i) $f(x, y, x) \geqslant 0$ for all $x \in K$ and $y \in T(x)$.

(ii) f is of class *(SPM)*.

(iii) T is upper semicontinuous, and $T(x)$ is nonempty and convex for every $x \in K$.

(iv) For every $y \in D$, the bifunction $(x, u) \longmapsto f(x, y, u)$ is 0-diagonally convex on $K \times K$.

(v) For every $u \in K$, the bifunction $(x, y) \longmapsto f(x, y, u)$ is upper semi-continuous on $\mathrm{co}(E) \times D$ for every nonempty finite set $E \subset K$.

(vi) *(Coercivity)* There is a nonempty compact set $K_0 \subset K$, and there is a nonempty compact convex set $K_1 \subset K$ such that

$$x \in K \cap K_0^c \Longrightarrow f(x, y, u_x) < 0 \quad \text{for some } u_x \in K_1 \text{ and for all } y \in T(x).$$

Then there exist $\widehat{x}\in K$ and $\widehat{y}\in T(\widehat{x})$ such that

$$f(\widehat{x},\widehat{y},u)\geqslant 0 \quad \text{for all } u\in K. \tag{1}$$

We note that if the function f given above is continuous on $K\times D\times K$, then conditions (ii) and (v) are automatically satisfied. We remark that most of the results in the literature concerning the existence of the problem (1) require the continuity of the function f and of course some other conditions which are different from the corresponding conditions of Corollary 4.1. See, e.g., Yao (1991) Theorem 3.1 and Cubiotti and Yao (1997) Corollary 3.6 for corresponding results in quasi-case.

In preparation of an existence result for a problem of class (GPM), we show:

LEMMA 4.2. Let X and Y be real Hausdorff topological vector spaces, let $K\subset X$ be nonempty and convex, let $D\subset Y$ be nonemtpy, let $T:K\longrightarrow 2^D$ be a multi-valued mapping, and let $f:K\times D\times K\longrightarrow \mathcal{Z}$ be a trifunction of class *(GPM)* with respect to T. If $u\in K$, and $\{(x_\alpha,y_\alpha)\}_{\alpha\in I}$ is a net in $K\times D$ satisfying $y_\alpha\in T(x_\alpha)$, $x_\alpha\to x\in K$, and

$$f(x_\alpha,y_\alpha,tu+(1-t)x)\in(-\text{Int}C)^c \quad \text{for all } \alpha\in I \text{ and for all } 0\leqslant t\leqslant 1,$$

then there exists $y_u\in T(x)$ such that $f(x,y_u,u)\in(-\text{Int}C)^c$.

Proof. For every $u\in K$ and for every $\alpha\in I$, let $A_\alpha(u)$ be given as in the proof of Lemma 4.1. Note that $A_\alpha(u)\subset(-\text{Int}C)^c$ and $A_\alpha(x)\subset(-\text{Int}C)^c$ for all $\alpha\in I$.

Suppose to the contrary that $f(x,y,u)\in(-\text{Int}C)$ for all $y\in \underline{T(x).}$ Let $w\in \text{Int}C$ be arbitrary. Since $A_\alpha(x)\subset(-\text{Int}C)^c\subset(-w-\text{Int}C)^c$, then $\overline{A_\alpha(x)}\cap(-w-\text{Int}C)=\emptyset$. By the definition of the class of (GPM), there exists $\alpha_0\in I$ and $y_u\in T(x)$ such that

$$\overline{A_\alpha(u)}\subset f(x,y_u,u)-w-\text{Int}C\subset(-\text{Int}C) \quad \text{for all } \alpha\geqslant\alpha_0,$$

which is a contradiction. The proof is now complete. □

The following result is then a consequence of Theorem 3.3 and Lemma 4.2.

THEOREM 4.2. Let X and Y be real Hausdorff topological vector spaces. Let $D\subset Y$ be nonempty, and let $K\subset X$ be nonempty and convex. Let $T:K\longrightarrow 2^D$ be a multi-valued mapping, and let $f:K\times D\times K\longrightarrow \mathcal{Z}$ be a trifunction. Assume that the following conditions are satisfied.

(i) T is upper semicontinuous, and $T(x)$ is nonempty and convex for every $x\in K$.

(ii) f is of class *(GPM)* with respect to T.

(iii) For every $x \in K$, there exists $y \in T(x)$ such that $f(x, y, x) = 0$.

(iv) For each fixed $u \in K$, the bifunction $(x, y) \longmapsto f(x, y, u)$ is C-upper semicontinuous on $K \times D$.

(v) For every fixed $(x, y) \in K \times D$, the function $u \longmapsto f(x, y, u)$ is C-convex.

(vi) For every fixed $(x, u) \in K \times K$, the function $y \longmapsto f(x, y, u)$ is C-upper semicontinuous and C-concave.

(vii) There is a nonempty compact set $K_0 \subset K$, and there is a nonempty compact convex set $K_1 \subset K$ such that if $x \in K \cap K_0^c$, then $f(x, y, u_x) \in (-\text{Int}C)$ for some $u_x \in K_1$ and all $y \in T(x)$.

Then there exist $\widehat{x} \in K$ and $\widehat{y} \in T(\widehat{x})$ such that $f(\widehat{x}, \widehat{y}, u) \in (-\text{Int}C)^c$ for all $u \in K$.

In Theorem 4.2, if $Z = \mathbb{R}$ and $C = \mathbb{R}^+$, then we obtain the following existence result for scalar generalized equilibrium problem, from which we can deduce and extend some results on existence of solutions for scalar variational inequalities associated with pseudomonotone multi-valued operators in the sense of Browder.

THEOREM 4.3. *Let X and Y be real Hausdorff topological vector spaces. Let $D \subset Y$ be nonempty, and let $K \subset X$ be nonempty and convex. Let $T : K \longrightarrow 2^D$ be a multi-valued mapping, and let $f : K \times D \times K \longrightarrow \mathbb{R}$ be a trifunction. Assume that the following conditions are satisfied.*

(i) *T is upper semicontinuous on K, and $T(x)$ is nonempty and convex for every $x \in K$.*

(ii) *f is of class (GPM) with respect to T.*

(iii) *For every $x \in K$, there exists $y \in T(x)$ such that $f(x, y, x) = 0$.*

(iv) *For every fixed $u \in K$, the bifunction $(x, y) \longmapsto f(x, y, u)$ is C-upper semicontinuous on $K \times D$.*

(v) *For every fixed $(x, y) \in K \times D$, the function $u \longmapsto f(x, y, u)$ is C-convex.*

(vi) *For every fixed $(x, u) \in K \times K$, the function $y \longmapsto f(x, y, u)$ is C-upper semicontinuous and C-concave.*

(vii) *There is a nonempty compact set* $K_0 \subset K$*, and there is a nonempty compact convex set* $K_1 \subset K$ *such that*

$$x \in K \cap K_0^c \Longrightarrow f(x, y, u_x) < 0 \quad \text{for some } u_x \in K_1 \text{ and all } y \in T(x).$$

Then there exist $\widehat{x} \in K$ *and* $\widehat{y} \in T\widehat{x})$ *such that* $f(\widehat{x}, \widehat{y}, u) \geqslant 0$ *for all* $u \in K$.

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